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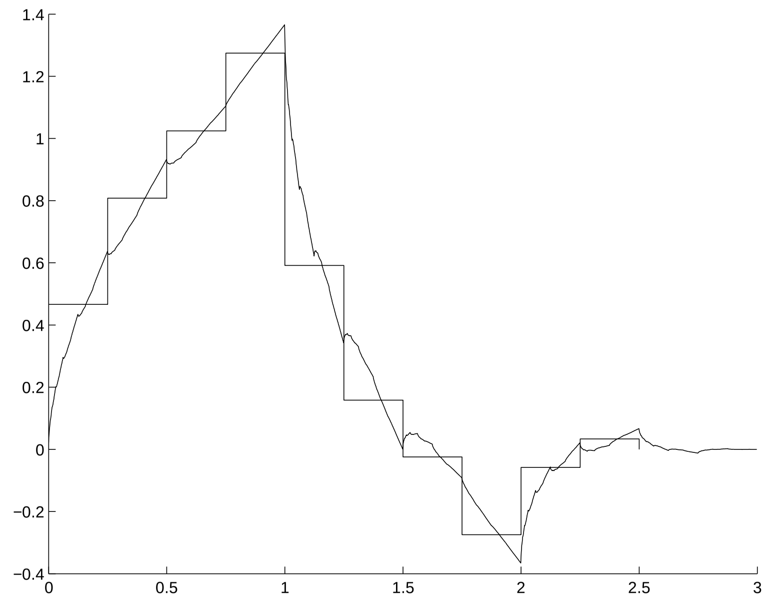
WAVELETS

from MATH to PRACTICE

Geometry and Visualization, April 19–25, 2008

Content



Fourier analysis (Joseph Fourier, 1807)

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx) = \sum_{k=-\infty}^{\infty} c_k e^{\imath kx}$$

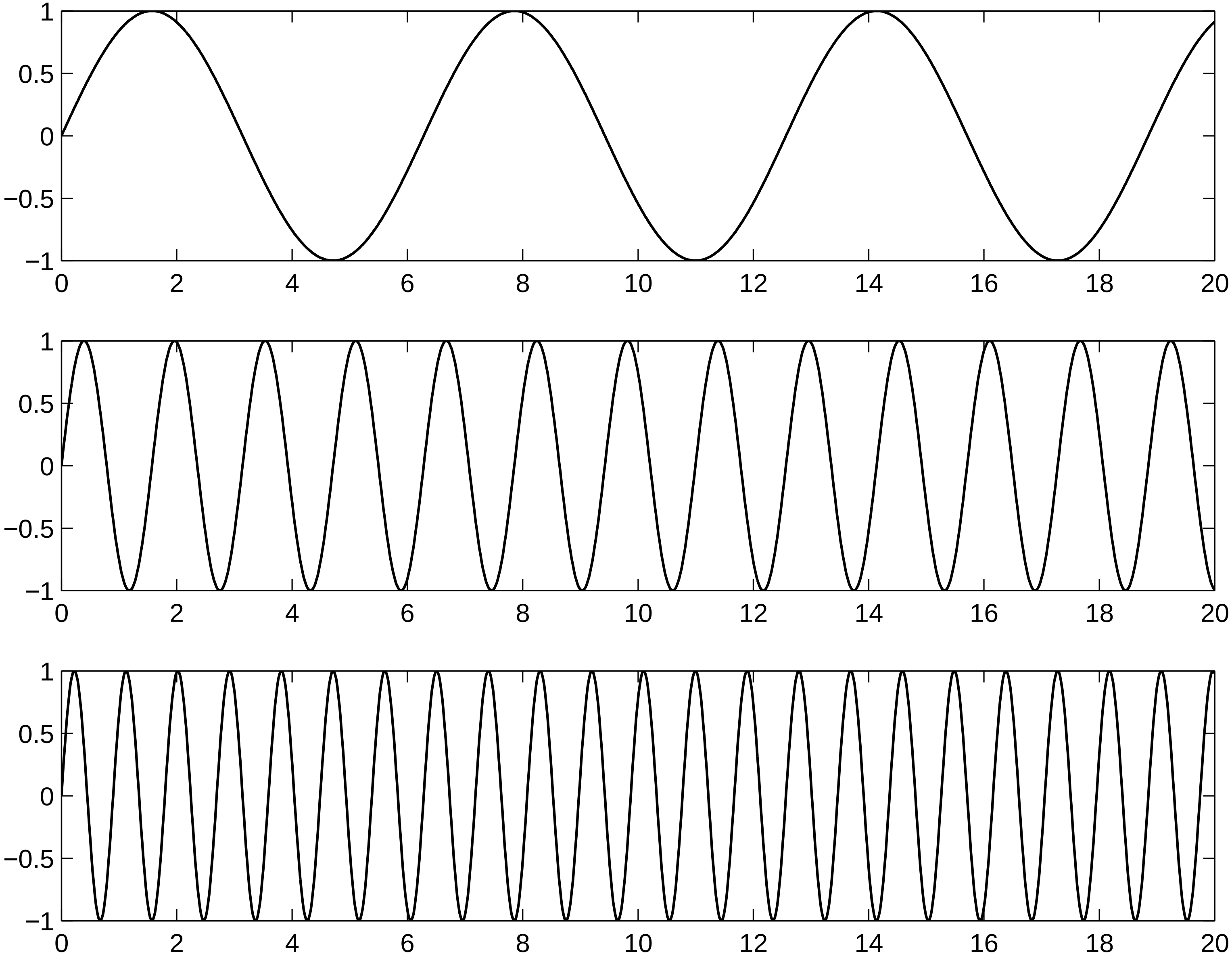
$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos kx \, dx$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin kx \, dx$$

$$c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \, e^{-\imath kx} \, dx$$

Parseval equality – energy of function f

$$\|f\|^2 = \int_{-\pi}^{\pi} |f(x)|^2 \, dx = \sum_{k=0}^{\infty} (|a_k|^2 + |b_k|^2) = \sum_{k=-\infty}^{\infty} |c_k|^2$$


1
0.5
0
−0.5
−1
0
2
4
6
8
10
12
14
16
18
20

Advantages of Fourier analysis

- $\{e^{\imath kx}\}_k$ is an orthogonal function system

$$(e^{\imath kx},\, e^{\imath lx}) = \int_{-\pi}^{\pi} e^{\imath kx} e^{-\imath lx}\, dx = \begin{cases} 0, & \text{za } k \neq l, \\ 2\pi, & \text{za } k = l, \end{cases}$$

- $e^{\imath kx}$ are eigenfunctions of the differential and difference operators

$$\frac{d}{dx} e^{\imath kx} = \imath k\, e^{\imath kx}, \qquad \Delta\, e^{\imath kx} = \left(\frac{e^{\imath kh} - 1}{h} \right) e^{\imath kx}$$

- FFT algorithm : $\mathbf{y} = \mathcal{F}_N\, \mathbf{x}$, complexity $O(N \log_2 N)$

$$\begin{aligned} y_j &= y_j^e + W_N^j y_j^o \\ y_{M+j} &= y_j^e - W_N^j y_j^o \end{aligned} \qquad j = 0, \ldots, M-1, \qquad \begin{pmatrix} W_N = e^{\imath 2\pi/N} \\ N = 2M \end{pmatrix}$$

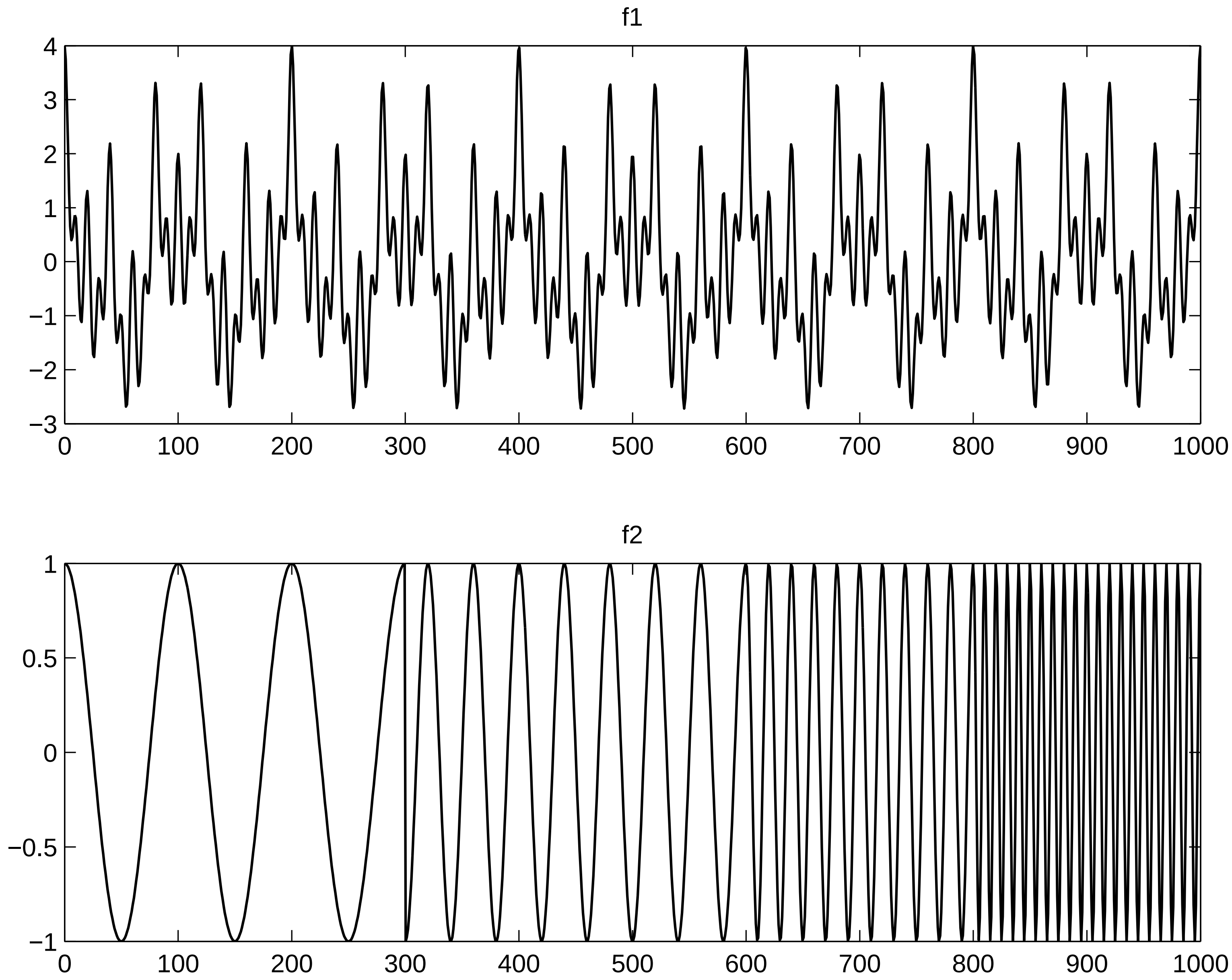
Disadvantages of Fourier analysis

steady signal – frequency content does not change in time

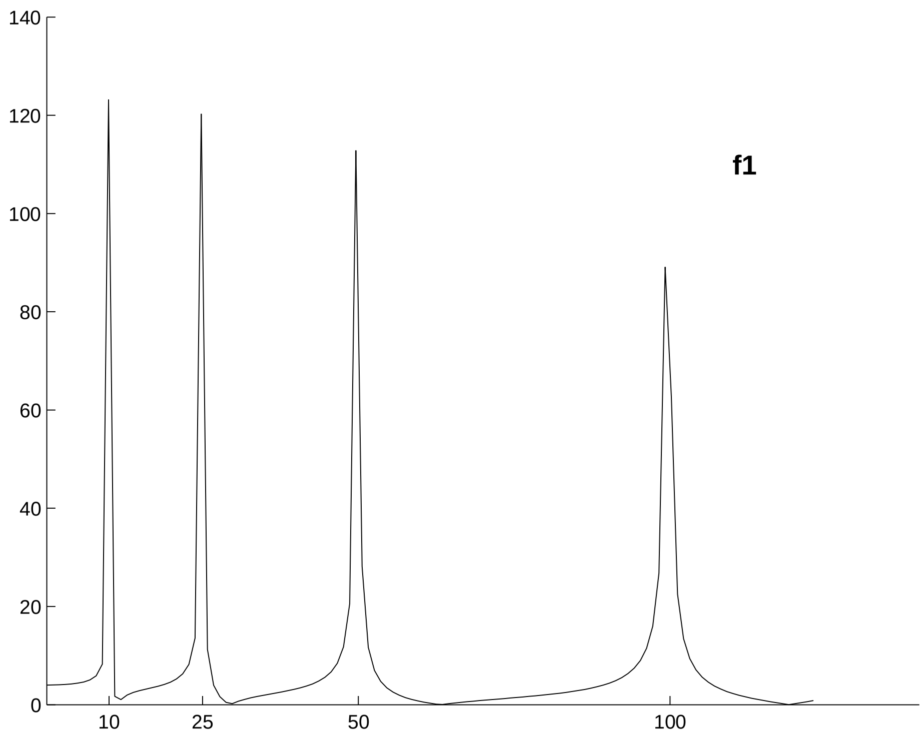
$$f1(x) = \cos(2\pi * 10 * x) + \cos(2\pi * 25 * x) + \cos(2\pi * 50 * x) + \cos(2\pi * 100 * x)$$

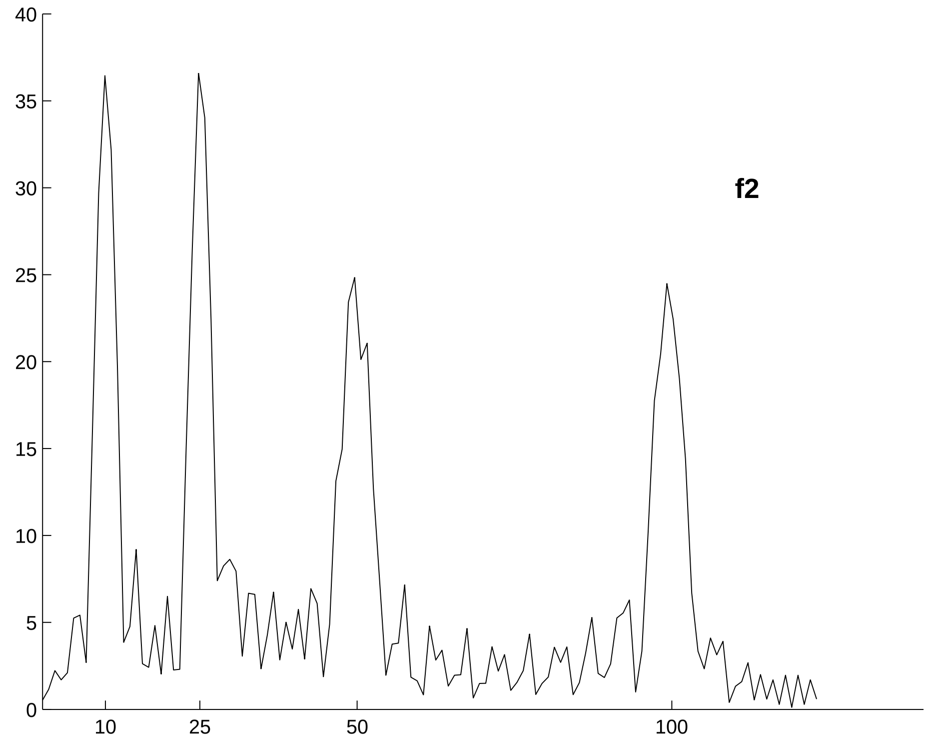
unsteady signal – frequency content changes in time

$$f2(x) = \begin{cases} \cos(2\pi * 10 * x), & 0 < x < 300 \\ \cos(2\pi * 25 * x), & 300 < x < 600 \\ \cos(2\pi * 50 * x), & 600 < x < 800 \\ \cos(2\pi * 100 * x), & 800 < x < 1000 \end{cases}$$


f1
4
3
2
1
0
−1
−2
−3
0 100 200 300 400 500 600 700 800 900 1000
f2
1
0.5
0
−0.5
−1
0 100 200 300 400 500 600 700 800 900 1000

Fourier specters





Transform is defined by the inner product $(f,\, g)$

Fourier $$\widehat{f}(\omega) = \int_{-\infty}^{\infty} f(x)\, e^{-\imath x \omega}\, dx$$

Short Time Fourier $$STFT_f(\omega, \tau) = \widehat{f}(\omega), \quad x \in [\tau, \tau + 1]$$

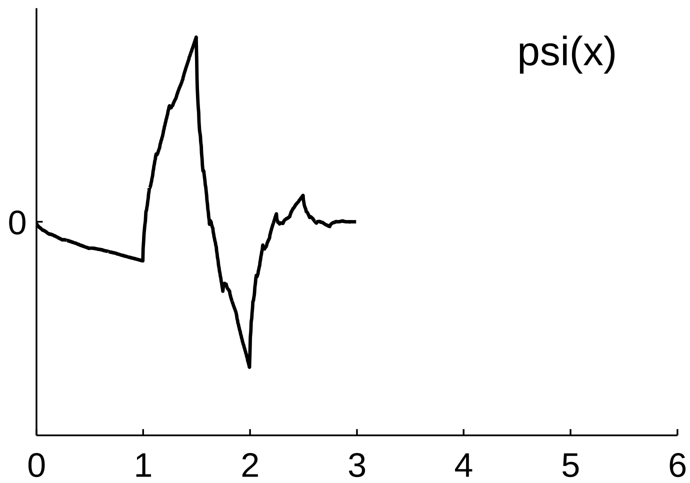
Fourier transform of the function $f(x)\, W(x - \tau), \quad W(x) = \begin{cases} 1, & x \in [0,1) \\ 0, & x \notin [0,1) \end{cases}$

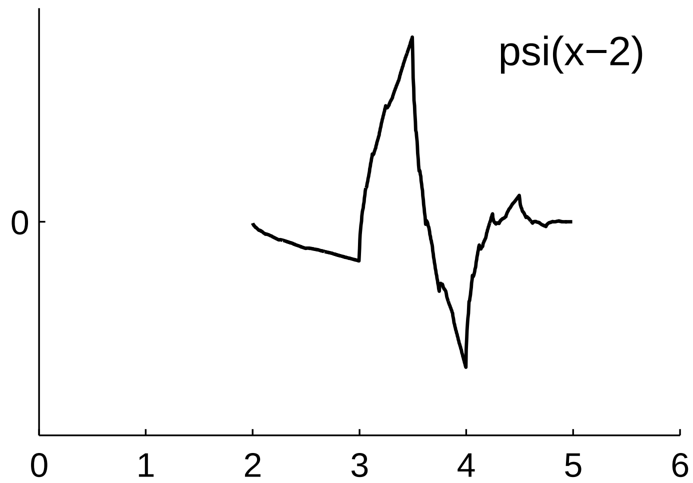
Wavelet $$WT_f(a, b) = \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} f(x)\, \overline{\psi}(\frac{x-b}{a})\, dx$$

Wavelet is an oscillatory function with a compact support

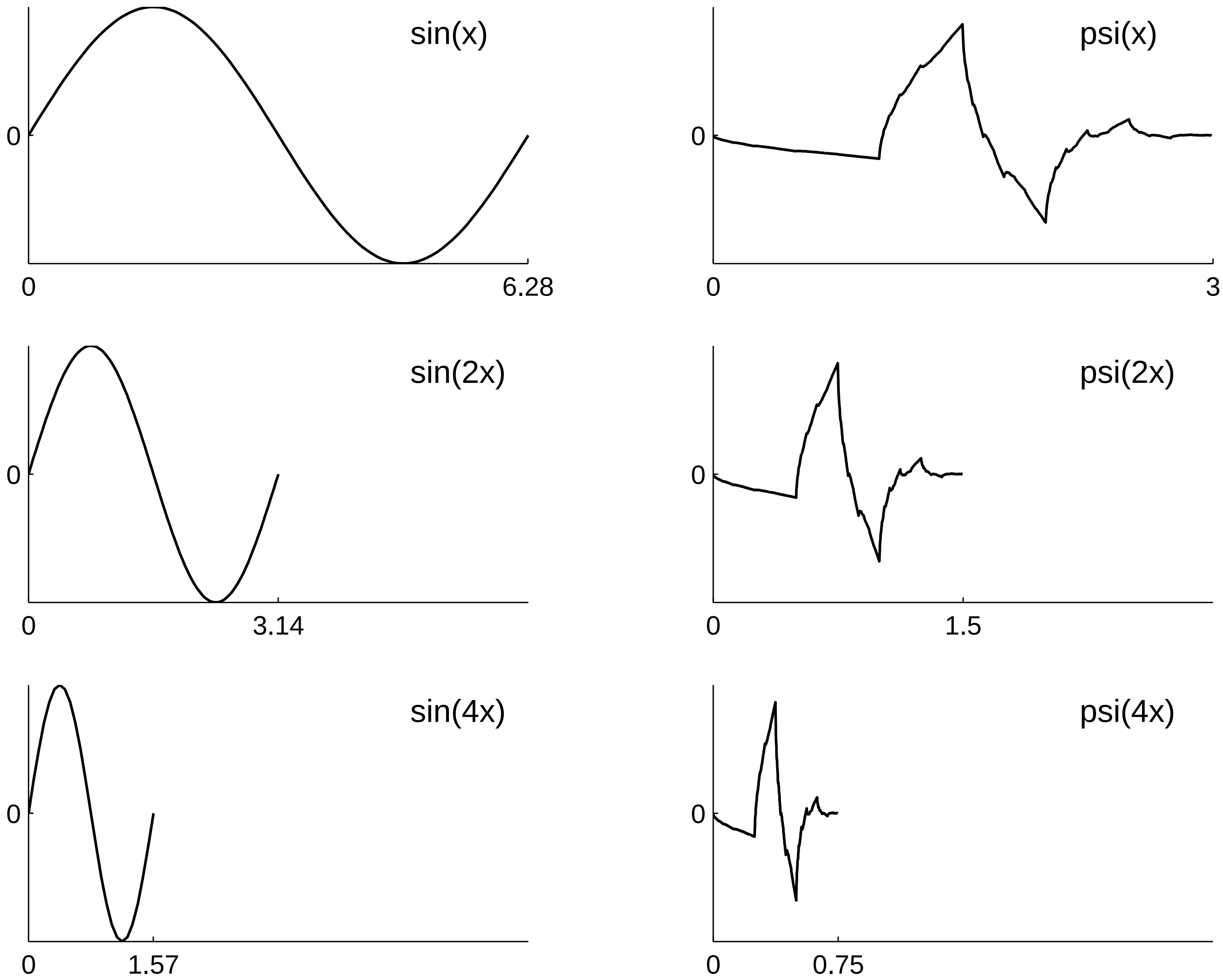
$$\psi_{a,b}(x) = \frac{1}{\sqrt{a}}\psi\left(\frac{x-b}{a}\right), \qquad \int \psi(x)\,dx = 0, \quad \psi(x) = 0,\, x \notin [0,\, N-1]$$

Translation (b) – time resolution

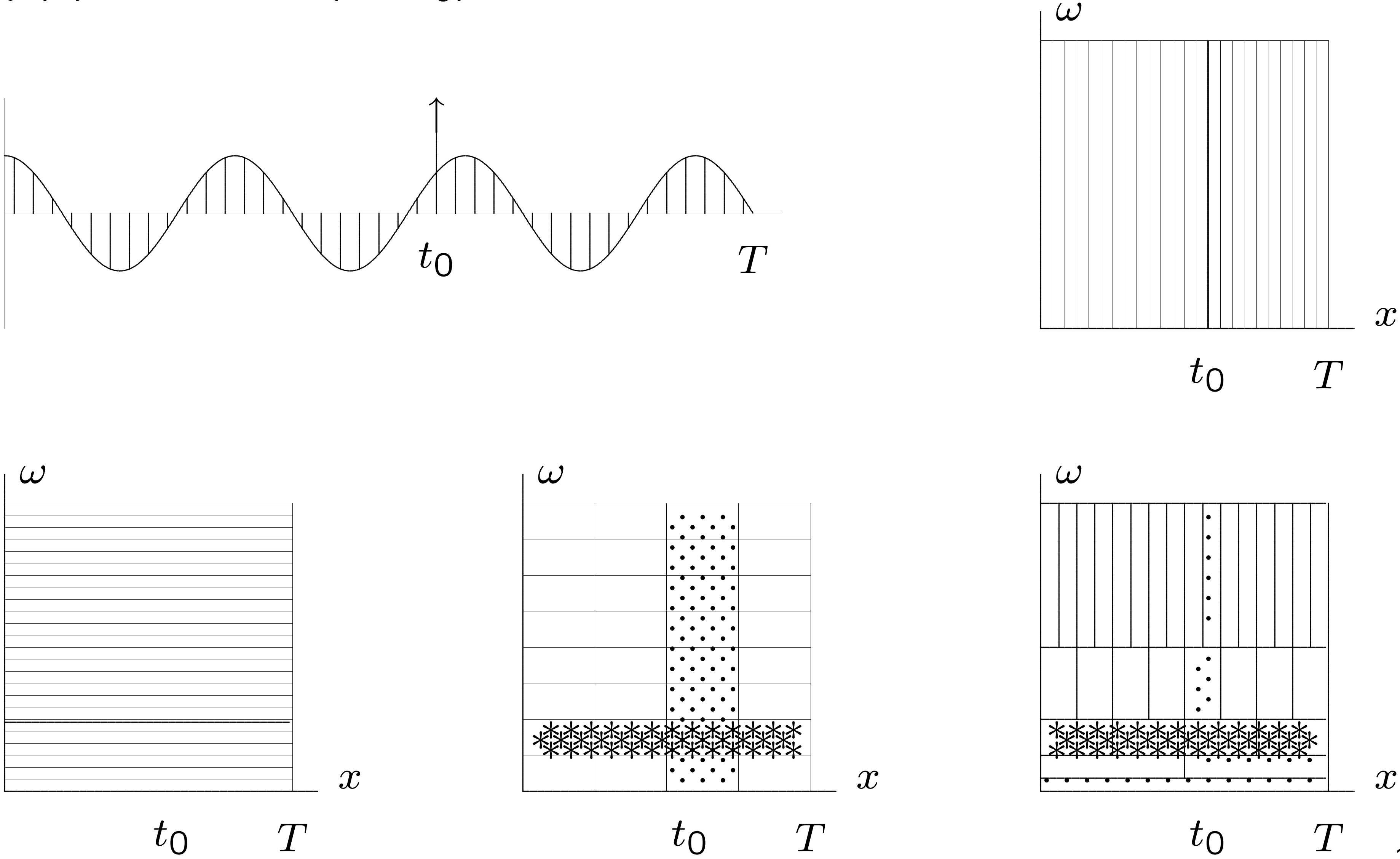




Dilatation (a) – frequency resolution

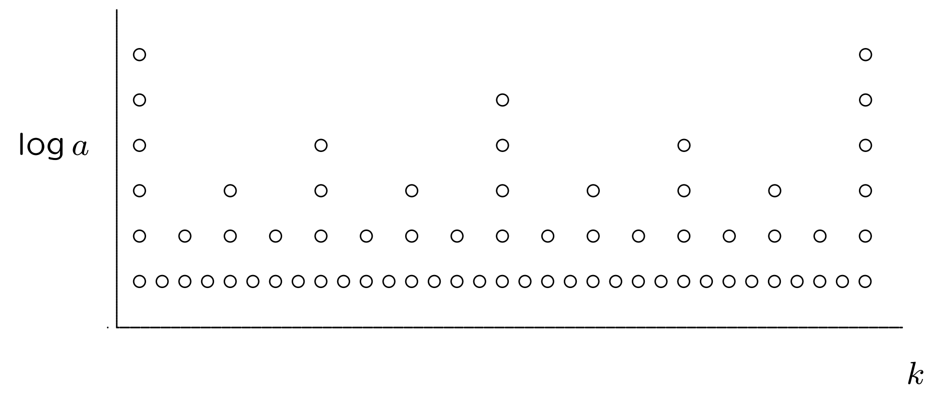

sin(x)
0
0
6.28
psi(x)
0
0
3
sin(2x)
0
0
3.14
psi(2x)
0
0
1.5
sin(4x)
0
0
1.57
psi(4x)
0
0
0.75

$$f(x) = \sin x + \delta(x - t_0)$$



Discrete wavelets

Diadic sampling $a = 2^j,\ b = k\,2^j$



$$\psi_{jk}(x) = 2^{-j/2}\psi(2^{-j}x - k)$$

$$\psi_{jk}(x) \neq 0, \quad x \in [2^j k, 2^j(k+1)]$$

Multiresolution

$$f(x) = \sum_{j\in\mathcal{Z}} \sum_{k\in\mathcal{Z}} b_{j,k}\,\psi_{j,k}(x) \qquad f(x) \approx \sum_{k\in\mathcal{Z}} a_{J,k}\,\varphi_{J,k}(x) + \sum_{l=j}^{J} \sum_{k\in\mathcal{Z}} b_{l,k}\,\psi_{l,k}(x)$$

$$\mathcal{L}_2(R) = \sum_{j=-\infty}^{\infty} \mathcal{W}_j, \qquad \mathcal{V}_{j-1} = \mathcal{V}_J \oplus \mathcal{W}_J \oplus \mathcal{W}_{J-1} \oplus \cdots \oplus \mathcal{W}_j, \quad J > j$$

Multiresolution analysis is the decomposition of Hilbert's space $\mathcal{L}_2(R)$ on a series of closed subspaces $\{\mathcal{V}_j\}_{j\in Z}$ such that

(1) $$\cdots \subset \mathcal{V}_2 \subset \mathcal{V}_1 \subset \mathcal{V}_0 \subset \mathcal{V}_{-1} \subset \mathcal{V}_{-2} \subset \ldots$$

(2) $$\cap_{j\in Z}\mathcal{V}_j = \{0\}, \qquad \overline{\cup_{j\in Z}\mathcal{V}_j} = \mathcal{L}_2(R)$$

(3) $$\forall f \in \mathcal{L}_2(R) \ \text{ i } \ \forall j \in Z, \qquad f(x) \in \mathcal{V}_j \quad \Longleftrightarrow \quad f(2x) \in \mathcal{V}_{j-1}$$

(4) $$\forall f \in \mathcal{L}_2(R) \ \text{ i } \ \forall k \in Z, \qquad f(x) \in \mathcal{V}_0 \quad \Longleftrightarrow \quad f(x-k) \in \mathcal{V}_0$$

(5) $\exists \varphi \in \mathcal{V}_0$ so that $\{\varphi(x-k)\}_{k\in Z}$ is Riesz's basis of subspace $\mathcal{V}_0$

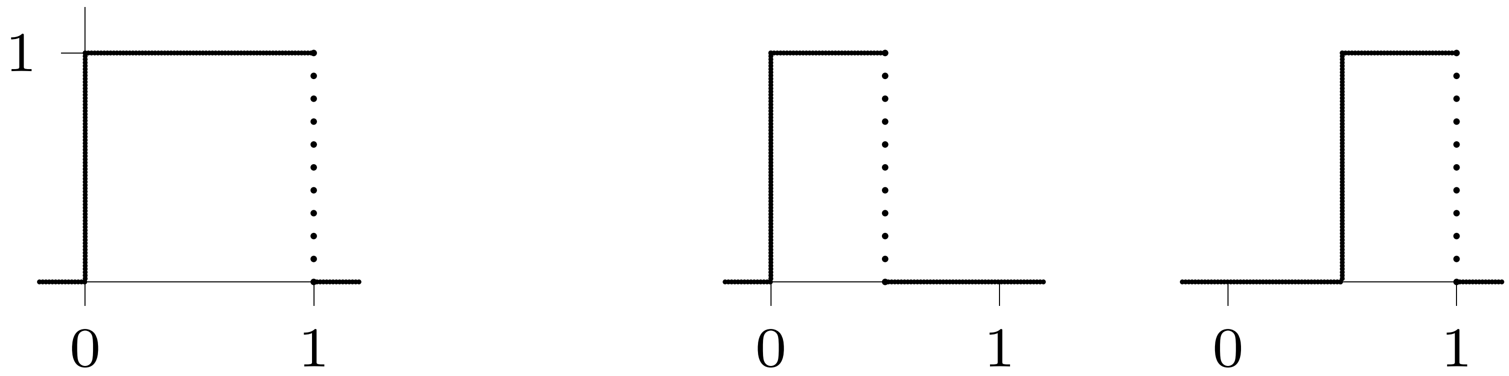
$\varphi_{j,k}(x) \equiv 2^{-j/2}\varphi(2^{-j}x-k), \quad k \in \mathcal{Z},$ are basis functions of subspace $\mathcal{V}_j$

Scaling function $\varphi(x)$

$$\mathcal{V}_0 \subset \mathcal{V}_{-1} \quad \longrightarrow \quad \varphi(x) = \sum_{k \in \mathcal{Z}} c(k)\,\sqrt{2}\varphi(2x-k)$$ *dilatation equation*

- box function $\quad c(0) = c(1) = \frac{1}{\sqrt{2}}$

$$\varphi(x) \quad = \quad \varphi(2x) \quad + \quad \varphi(2x-1)$$

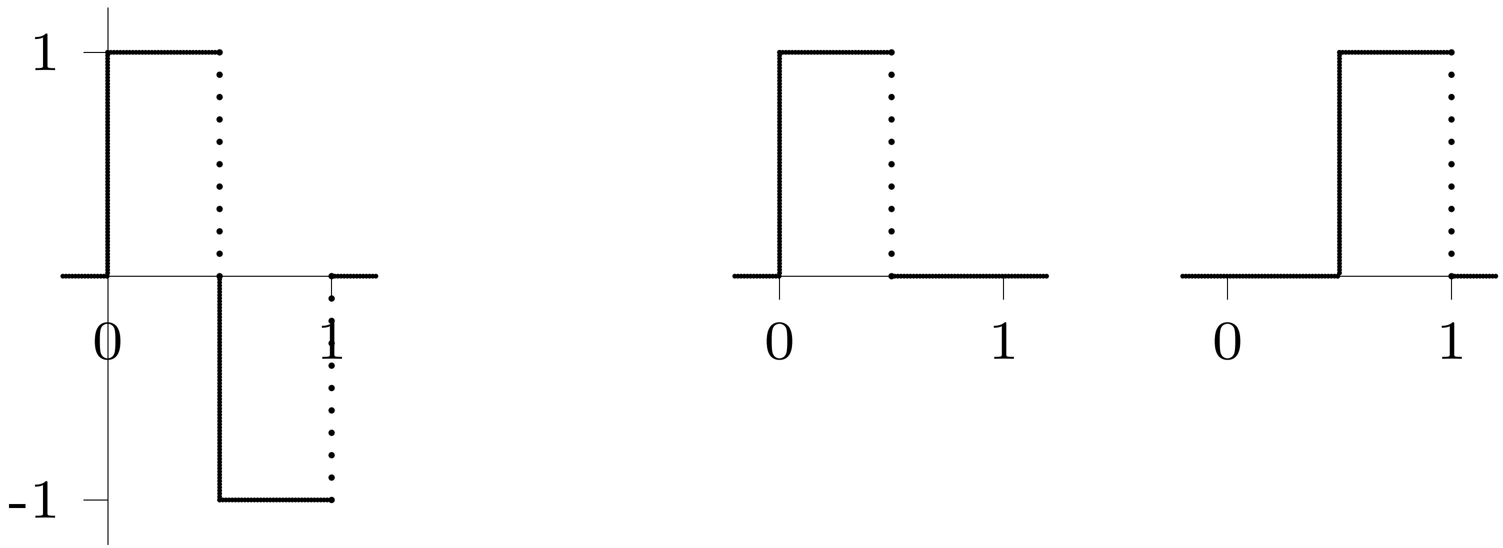


Wavelet $\psi(x)$

$$\mathcal{V}_0 \oplus \mathcal{W}_0 = \mathcal{V}_{-1} \quad \longrightarrow \quad \psi(x) = \sum_{k \in \mathcal{Z}} d(k)\,\sqrt{2}\varphi(2x-k)$$ *wavelet equation*

- Haar wavelet (1909) $\quad d(0) = 1/\sqrt{2}, \quad d(1) = -1/\sqrt{2}$

$$\psi(x) \quad = \quad \varphi(2x) \quad - \quad \varphi(2x-1)$$



applet

Solving dilatation equation

Recursion $\quad \Phi(0) = \big(\varphi(0), \dots, \varphi(N-2)\big)^T$, $M_0 = \{\sqrt{2}\, c(2i-j)\}$, $M_1 = \{\sqrt{2}\, c(2i-j+1)\}$

$$\Phi(0) = M_0\, \Phi(0), \quad \Phi\big(\tfrac{1}{2}\big) = M_1\, \Phi(0), \quad \Phi\big(\tfrac{1}{4}\big) = M_0\, \Phi\big(\tfrac{1}{2}\big), \dots$$

Cascade algorithm

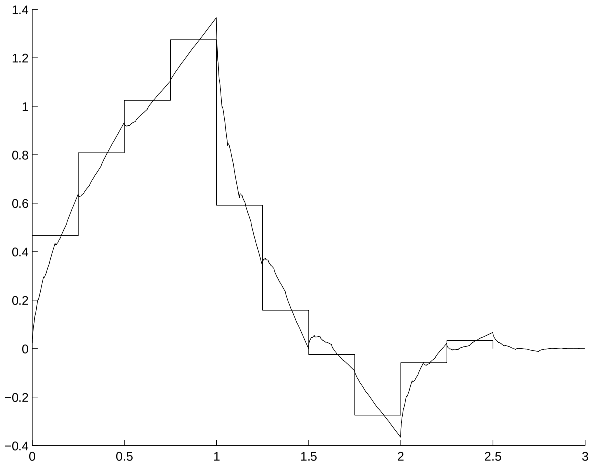
$$\varphi^{(0)}(x) \text{ box f.}, \qquad \varphi^{(j+1)}(x) = \sum_{k=0}^{N-1} c(k)\sqrt{2}\varphi^{(j)}(2x-k),\ j = 0, 1, \dots,$$

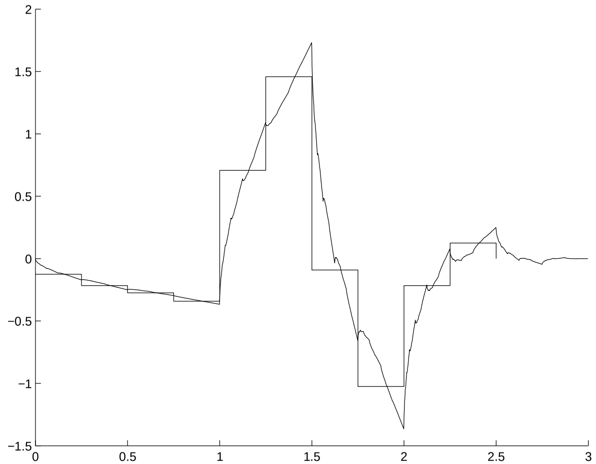
$$c(0) = \frac{1+\sqrt{3}}{4\sqrt{2}},$$

$$c(1) = \frac{3+\sqrt{3}}{4\sqrt{2}},$$

$$c(2) = \frac{3-\sqrt{3}}{4\sqrt{2}},$$

$$c(3) = \frac{1-\sqrt{3}}{4\sqrt{2}},$$





Pyramid algorithm (FWT complexity for orthogonal basis is $O(N)$)

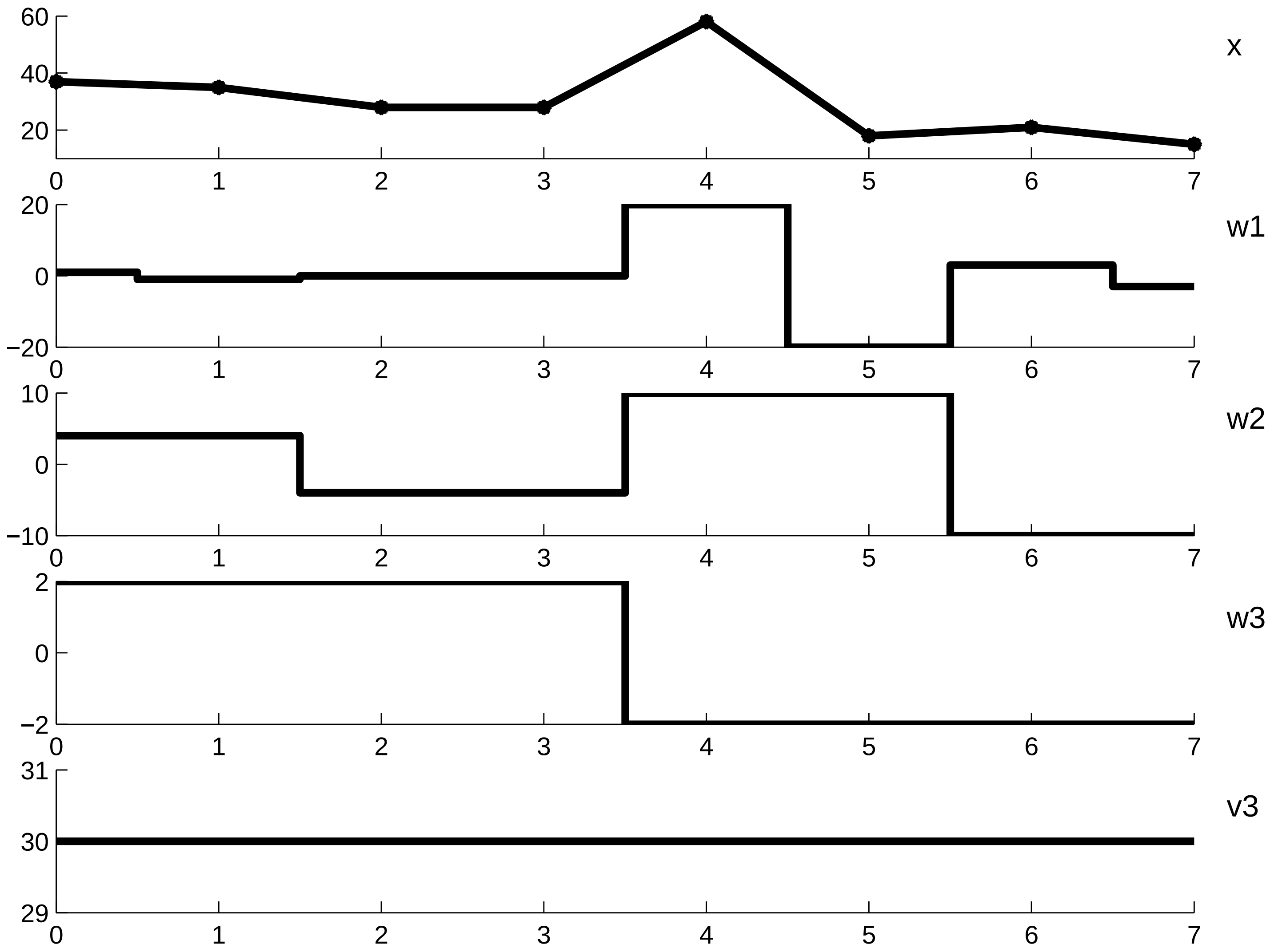
- $$f(x) \approx a_{3,0}\,\varphi_{3,0}(x) + \sum_{j=1}^{3} \sum_{k=0}^{2^{3-j}} b_{j,k}\,\psi_{j,k}(x), \qquad a_{j,k} = (f,\,\varphi_{j,k}), \quad b_{j,k} = (f,\,\psi_{j,k})$$

decomposition

$$a_{j,k} = \sum_l c(l-2k)a_{j-1,l}$$

$$b_{j,k} = \sum_l d(l-2k)a_{j-1,l}$$

37	35	28	28	58	18	21	15
36	28	38	18	1	0	20	3
32	28	4	10				
30	2						
30	2	4	10	1	0	20	3


x
w1
w2
w3
v3
60
40
20
0
−20
10
−10
2
−2
31
30
29
0 1 2 3 4 5 6 7

reconstruction $$a_{j-1,l} = \sum_k \big(c(l-2k)a_{j,k} + d(l-2k)b_{j,k}\big)$$

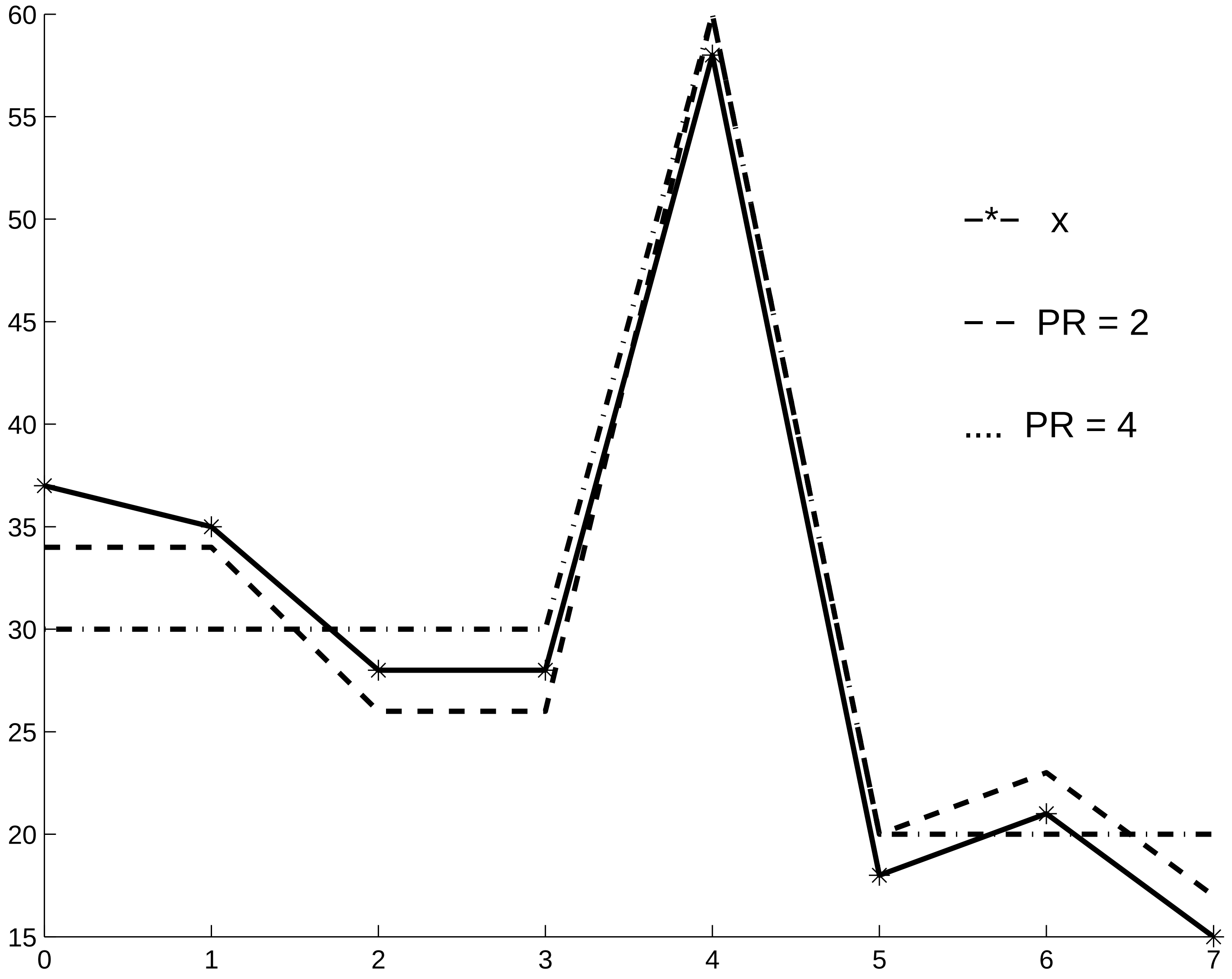
Compression

threshold $= 2$

30	2	4	10	1	0	20	3
30	0	4	10	0	0	20	3
30	30	4	10	0	0	20	3
34	26	40	20	0	0	20	3
34	34	26	26	60	20	23	17

threshold $= 4$

30	2	4	10	1	0	20	3
30	0	0	10	0	0	20	0
30	30	0	10	0	0	20	0
30	30	40	20	0	0	20	0
30	30	30	30	60	20	20	20


60
55
50
45
40
35
30
25
20
15
0
1
2
3
4
5
6
7
–*– x
– – PR = 2
.... PR = 4

Properties

Orthogonal basis $\{\varphi_{J,k},\ \psi_{j,k}\}$ defined by N nonzero coefficients $c(k)$, N even

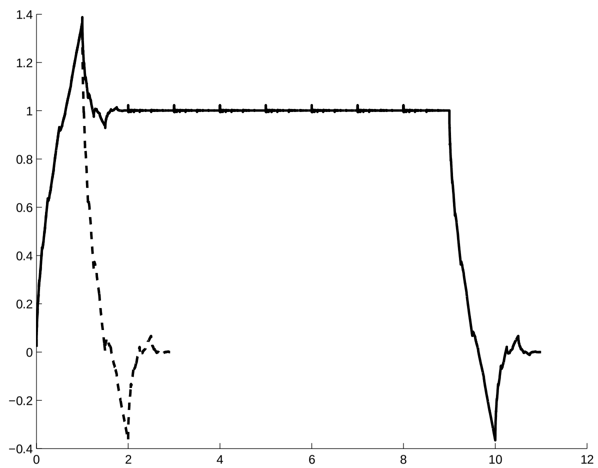
- Compact support is interval $[0,\ N-1]$
- Basis is orthogonal if for $k = 0, \ldots, N-1$,

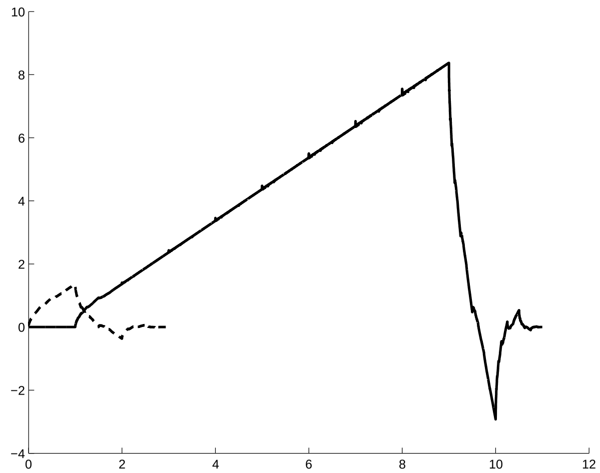
$$\sum_k c(k)c(k-2m) = \delta(m), \qquad d(k) = (-1)^k c(N-1-k)$$

- If $\omega = \pi$ is zero of order r ($N = 2\,r$) of function

$$\hat{c}(\omega) = \sum_{k=0}^{2r-1} c(k)\, e^{-\imath\omega k}$$

- polynomials x^m, $m = 0, \ldots, r-1$, can be reproduced by $\varphi(x-k)$, $k \in \mathcal{Z}$





- first r wavelet moments vanish

$$\int x^m \psi(x)\,dx = 0, \quad m = 0, \ldots, r-1,$$

- approximation error is

$$\|f - \sum_k a_{j,k}\varphi_{j,k}(x)\| \leq \text{const} \cdot 2^{jr}\|f^{(r)}\|$$

- wavelet coefficients decrease as

$$\int f(x)\psi(2^j x)\,dx \leq \text{const} \cdot 2^{-jr}$$

Analogy with filters

$$\mathbf{h} = \{h(n)\}, \qquad \widehat{h}(\omega) = \sum_n h(n)\, e^{-\imath n\omega} \qquad H(z) = \sum_n h(n)\, z^{-n} \quad (z = e^{\imath\omega})$$

$$y(n) = \sum_k h(k)\, x(n-k), \qquad \mathbf{y} = \mathbf{h} * \mathbf{x}, \qquad \widehat{y}(\omega) = \widehat{h}(\omega)\, \hat{\mathbf{x}}(\omega)$$

- Averaging filter $\quad \widehat{h}_0(0) = 1, \quad \widehat{h}_0(\pi) = 0 \qquad \text{lowpass}$

  box function $\quad \longleftrightarrow \quad y(n) = \frac{1}{2}\, x(n) + \frac{1}{2}\, x(n-1)$

- Differing filter $\quad \widehat{h}_1(0) = 0, \quad \widehat{h}_1(\pi) = 1 \qquad \text{highpass}$

  Haar wavelet $\quad \longleftrightarrow \quad y(n) = \frac{1}{2}\, x(n) - \frac{1}{2}\, x(n-1)$

- Downsampling $(\downarrow 2)$

  dilatation equation $\quad \longleftrightarrow \quad y(n) = \sum_k h(k)\, x(2n-k)$

Orthogonal filter bank is characterized by orthogonal matrices (pyramid alg.)

analysis

$$\mathbf{y}_1 = W_1\,\mathbf{x}, \qquad W_1 = \sqrt{2}\begin{pmatrix} 1/2 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/2 & 1/2 \\ 1/2 & -1/2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & -1/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/2 & -1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/2 & -1/2 \end{pmatrix}$$

syntesis

$$\mathbf{x} = W_1^\top\,\mathbf{y}_1, \qquad W_1^\top = \sqrt{2}\begin{pmatrix} 1/2 & 0 & 0 & 0 & 1/2 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & -1/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 0 & -1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 1/2 & 0 & 0 & 0 & -1/2 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 1/2 & 0 & 0 & 0 & -1/2 \end{pmatrix}$$

Fast Wavelet Transform FWT (complexity $O(N)$)

$$\mathbf{y}_1 = W_1 \mathbf{x} \qquad \mathbf{y}_2 = W_2 \mathbf{y}_1 \qquad \mathbf{y} = W_3 \mathbf{y}_2$$

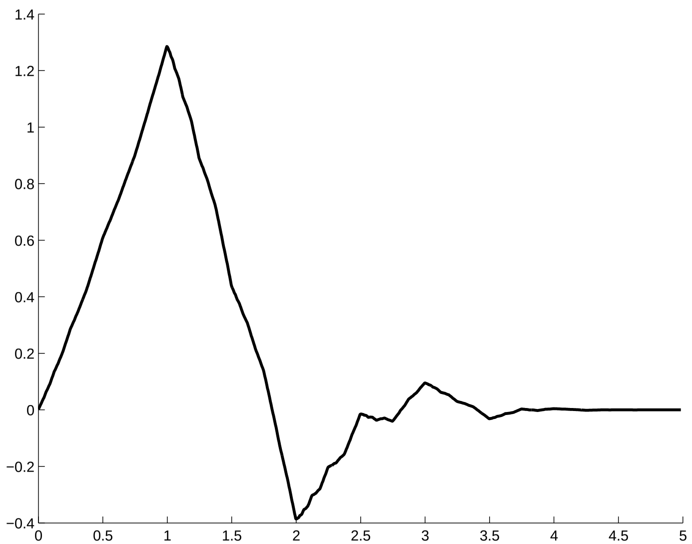
$$W_2 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad W_3 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

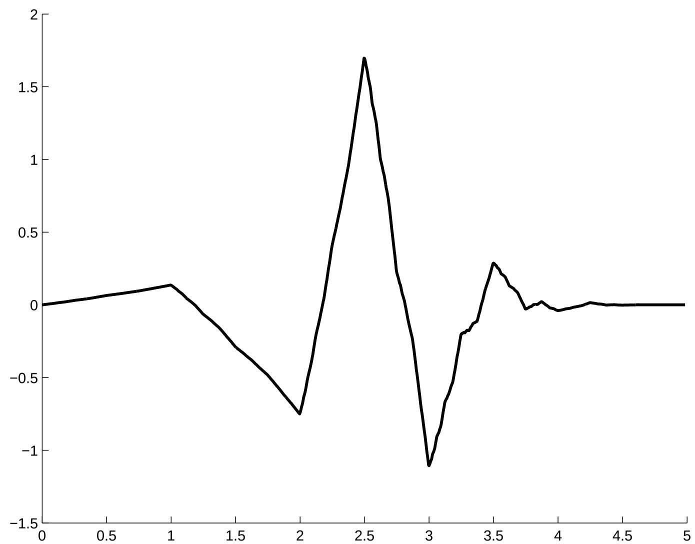
$$\mathbf{y} = \begin{pmatrix} a_{2,0} \\ b_{2,0} \\ b_{1,0} \\ b_{1,1} \\ b_{0,0} \\ b_{0,1} \\ b_{0,2} \\ b_{0,3} \end{pmatrix} = W \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{pmatrix}, \quad W = W_3 W_2 W_1 = 2\sqrt{2} \begin{pmatrix} \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} \\ \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} \\ \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} \end{pmatrix}$$

Daubechies wavelets Dbr (Ingrid Daubechies, 1988)

- have no explicite expression, except for Haar wavelet Db*1*,
- have compact support $[0, 2r-1]$,
- make orthonormal basis (efficent computation – pyramid algorithm, FWT),
- reproduce polynomials of order $r-1$ (nice approximation properties),
- have r vanishing moments (nice compression properties),
- belong to the class $\mathcal{C}^{\mu r}$, $\mu \approx 0.2$ for large r (certain smoothness).

Db3





frequency response

$$\widehat{c}(\omega) = \sum_k c(k) e^{-\imath k\omega}, \qquad C(z) = \sum_k c(k) z^{-k}, \quad (z = e^{\imath\omega})$$

power spectral response

$$P(z) = |C(z)|^2 = \sum_{n=-N+1}^{N-1} \left(\sum_{k=0}^{N-1} c(k)c(k-n) \right) z^{-n}$$

$$= 1 + \sum_{k=1}^{N/2} p(2k-1) \left(z^{-(2k-1)} + z^{2k-1} \right)$$

$$\widehat{p}(\omega) = |\widehat{c}(\omega)|^2 = 2 \left(\frac{1+\cos\omega}{2} \right)^r \sum_{k=0}^{r-1} \binom{r+k-1}{k} \left(\frac{1-\cos\omega}{2} \right)^k$$

averaging filter $\qquad \widehat{p}(\omega) = 1 + \cos\omega = \frac{1}{2}(1+e^{-\imath\omega})(1+e^{\imath\omega}) = |\widehat{c}(\omega)|^2$

Biorthogonal wavelets

Two sequences of multiresolution spaces

$$\mathcal{V}_j + \mathcal{W}_j = \mathcal{V}_{j-1}, \quad \tilde{\mathcal{V}}_j + \tilde{\mathcal{W}}_j = \tilde{\mathcal{V}}_{j-1}, \quad \longrightarrow \quad \mathcal{V}_j \perp \tilde{\mathcal{W}}_j, \quad \mathcal{W}_j \perp \tilde{\mathcal{V}}_j$$

Dilatation and wavelet equations

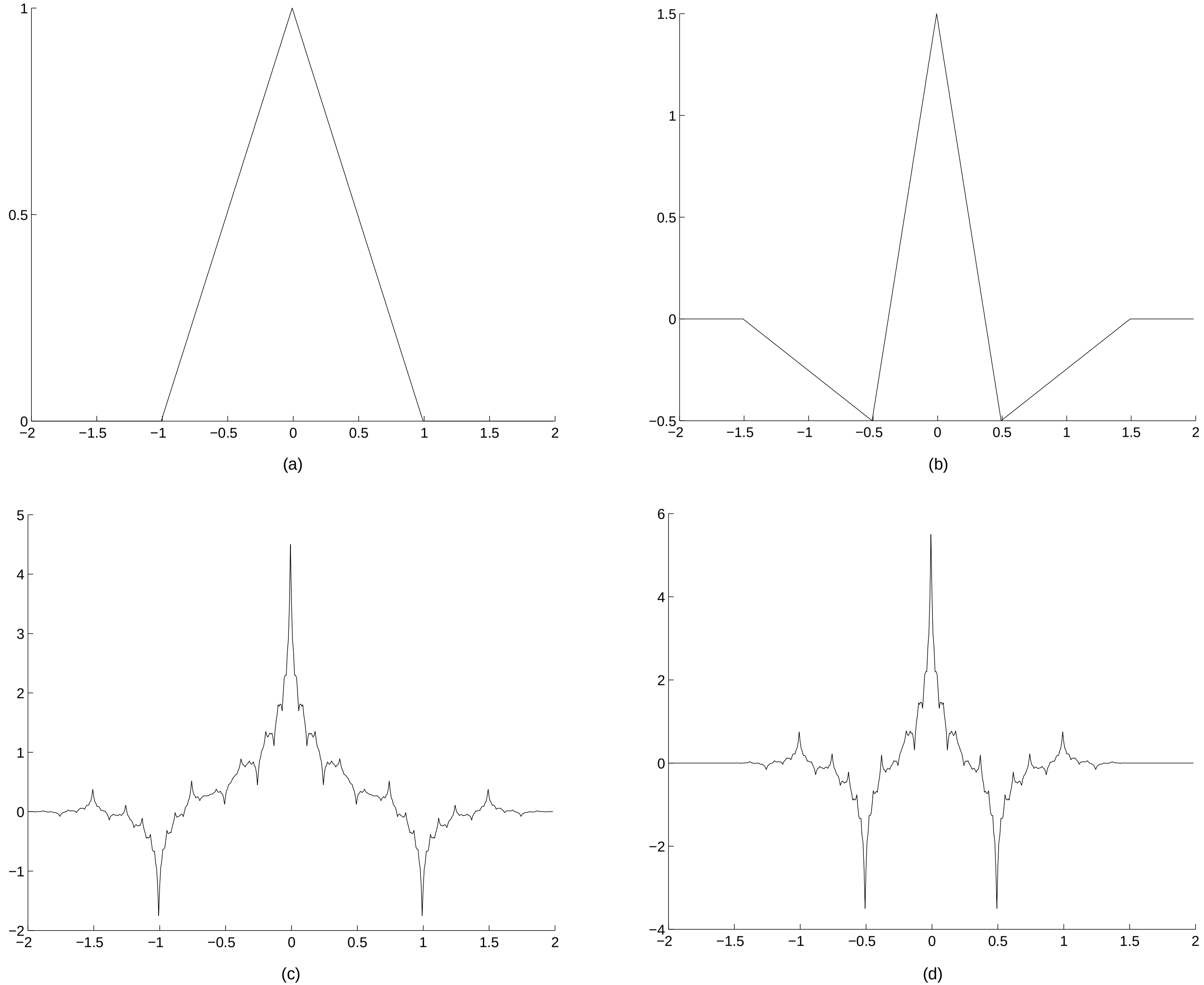
$$\varphi(x) = \sum_k h_0(k)\varphi(2x-k), \qquad \tilde{\varphi}(x) = 2\sum_k f_0(k)\tilde{\varphi}(2x-k),$$

$$\psi(x) = \sum_k h_1(k)\varphi(2x-k), \qquad \tilde{\psi}(x) = 2\sum_k f_1(k)\tilde{\varphi}(2x-k).$$

$$h_1(n) = (-1)^{n+1} f_0(1-n), \qquad f_1(n) = (-1)^{n+1} h_0(1-n), \qquad n = 0, \pm 1, \ldots.$$

$$(\varphi_{j,k}, \tilde{\psi}_{j,K}) = 0, \qquad (\tilde{\varphi}_{j,k}, \psi_{j,K}) = 0,$$

$$\left(\psi_{j,k}\,,\, \tilde{\psi}_{J,K}\right) = \delta(j-J)\,\delta(k-K), \qquad \left(\varphi_{j,k}\,,\, \tilde{\varphi}_{j,K}\right) = \delta(k-K).$$


(a)
(b)
(c)
(d)

Representation

$$g(x) = \sum_k a_{j,k}\tilde{\varphi}_{j,k}(x), \qquad a_{j,k} = \big(g,\, \varphi_{j,k}\big) = \int g(x)\varphi_{j,k}(x)\,dx,$$

$$g(x) = \sum_j \sum_k b_{j,k}\tilde{\psi}_{j,k}(x), \qquad b_{j,k} = \big(g,\, \psi_{j,k}\big) = \int g(x)\psi_{j,k}(x)\,dx.$$

Pyramid algorithm

analysis $$a_{j,k} = \sum_l h_0(l-2k)a_{j-1,l}, \qquad b_{j,k} = \sum_l h_1(l-2k)a_{j-1,l},$$

synthesis $$a_{j-1,l} = \sum_k \big(f_0(l-2k)a_{j,k} + f_1(l-2k)b_{j,k}\big)$$

$$\tilde{\psi} \in \mathcal{C}^m \qquad \longrightarrow \qquad \int x^k\psi(x)\,dt = 0,\ k = 0,\ldots,m,$$

Applications

Signal processing (analysis, synthesis, compression)

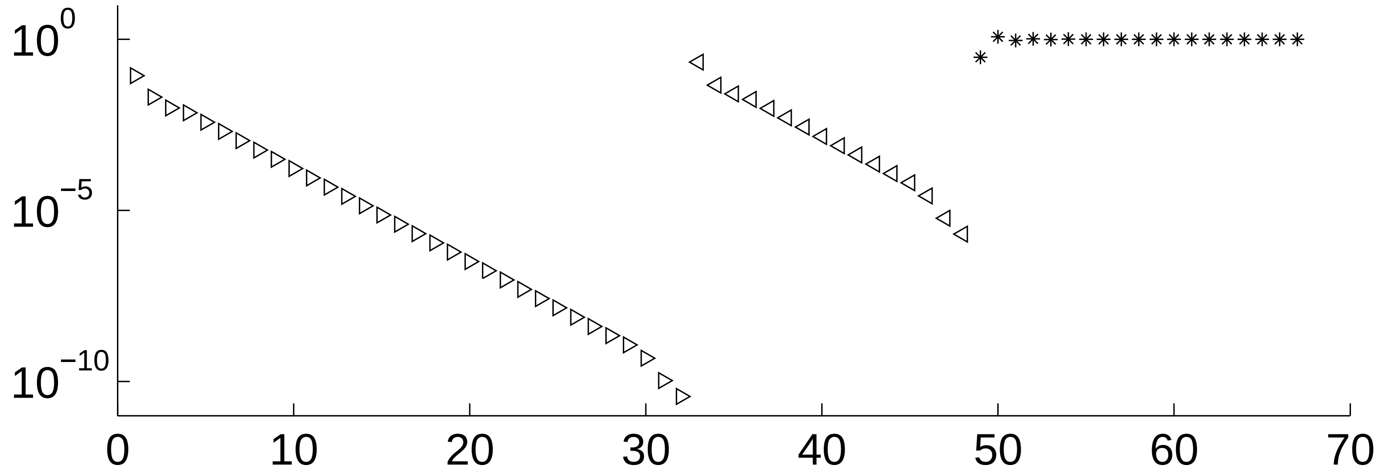
- Location and prediction of the earthquake.
- Study of distant galaxies.
- Analysis and compression of medical signals (ECG, EEG)
- Quality control by use of the sound signal analysis.
- Communications (compression)

Image processing

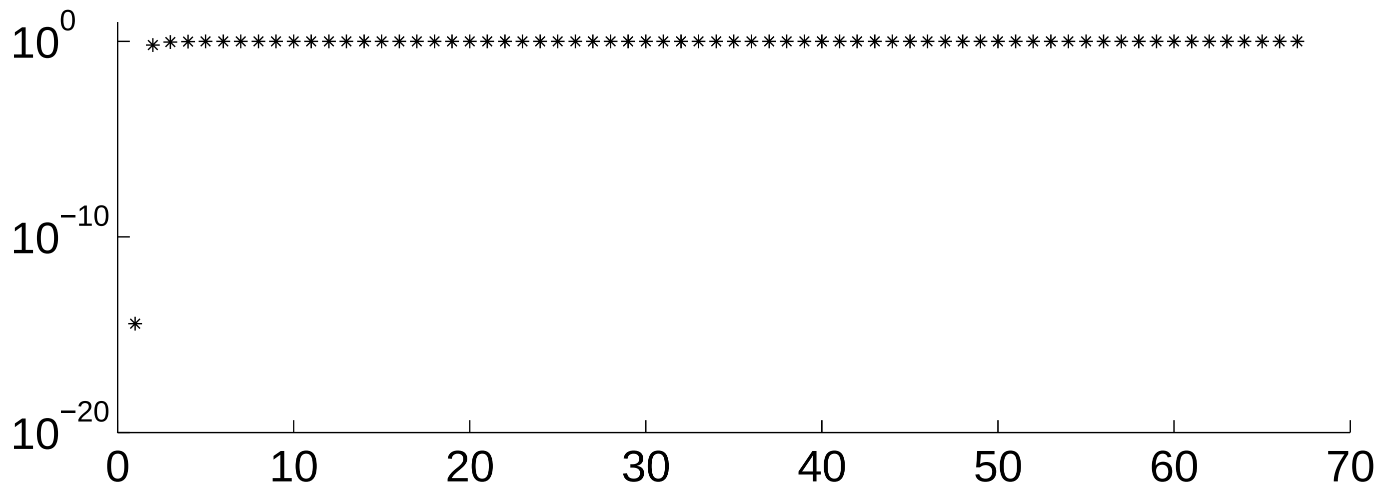
- Compression of finger prints in proportion 20:1 (JPEG 2000)
- Image compression
- Computer graphics (successive rendering)
- Computer vision (multiresolution approach)

Numerical modelling

Magnitude of wavelet (a) and spline (b) coefficients obtained by collocation method



(a)



(b)

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